

INTEGRATING INFINITESIMAL (SUPER) ACTIONS

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ABSTRACT. Let G be a (super) Lie group, \mathfrak{g} its (super) Lie algebra and let ρ be a representation of \mathfrak{g} as (smooth) vector fields on a (super) manifold M . We will show that there exists a smooth (left) action of G on M such that ρ is the map that associates the fundamental vector field on M to an algebra element if the following two conditions are satisfied: G is simply connected and all (smooth, even) vector fields $\rho(X)$ are complete. In the non-super case this proof reduces to a geometric proof of the result of Richard Palais.

1. INTRODUCTION

Let $\Phi : G \times M \rightarrow M$ be a smooth left-action of a Lie group G on a manifold M and let \mathfrak{g} be the Lie algebra of G . Associated to $X \in \mathfrak{g}$ we can define the fundamental vector field X^M on M by

$$X^M|_m = \frac{d}{dt}|_{t=0} \Phi(\exp(-tX), m) .$$

The map $X \mapsto X^M$ from the Lie algebra \mathfrak{g} to (smooth) vector fields on M is a Lie algebra homomorphism: $[X, Y]^M = [X^M, Y^M]$ (to have this without a minus sign we introduced the minus sign in the definition of the fundamental vector field X^M). Said differently, the map $X \mapsto X^M$ is a representation of \mathfrak{g} as vector fields on M . Conversely, one can ask the question whether a representation ρ of \mathfrak{g} as (smooth) vector fields on a manifold M (for each $X \in \mathfrak{g}$ we have a (smooth) vector field $\rho(X)$ on M) determines a left-action of G on M such that the representation is the one given by the fundamental vector fields, i.e., $\rho(X) = X^M$. Unfortunately, it is very easy to construct examples for which this is not possible (we will see several below). In order to find necessary and or sufficient conditions for this to be possible, we make some observations.

We start with a fixed $m_o \in M$ and we consider the graph of the map $g \mapsto \Phi(g, m_o)$, i.e., we consider the submanifold

$$L_{m_o} = \{ (g, m) \in G \times M \mid m = \Phi(g, m_o) \} \subset G \times M .$$

It is immediate that the tangent space to this submanifold is given by

$$T_{(g,m)}L_{m_o} = \{ X^r|_g - X^M|_m \mid X \in \mathfrak{g} \} ,$$

where X^r denotes the right-invariant vector field on G whose value at the identity $e \in G$ is X : $X^r|_e = X \in T_e G \cong \mathfrak{g}$. But we can say more: if we define the foliation $\mathcal{F} \subset T(G \times M)$ on $G \times M$ by

$$\mathcal{F}|_{(g,m)} = \{ X^r|_g - X^M|_m \mid X \in \mathfrak{g} \} ,$$

then L_{m_o} is a leaf, i.e., a maximal integral manifold, of \mathcal{F} . As the projection $p : G \times M \rightarrow G$ is a bijection from $L_{m_o} \rightarrow G$, we see that $\Phi(g, m_o)$ is completely determined by the leaf L_{m_o} of \mathcal{F} passing through $(e, m_o) \in G \times M$: $m = \Phi(g, m_o)$ is the unique element of M such that (e, m_o) and (g, m) lie on the same leaf of \mathcal{F} . It follows that the left-action of G on M is completely determined by the foliation \mathcal{F} on $G \times M$.

But we can say even more: if we denote by $\Phi_{\hat{X}}$ the flow of the vector field $\hat{X} = X^r - X^M$ on $G \times M$, then the definition of the fundamental vector field implies that it is given by

$$(1.1) \quad \Phi_{\hat{X}}(t, g, m) = (\exp(tX) \cdot g, \Phi(\exp(tX) \cdot g, m)) .$$

Said differently, the left-action (when restricted to elements of the form $\exp(X)$) can be expressed in terms of the flow of vector fields on $G \times M$, vector fields that depend only on the map from \mathfrak{g} to vector fields on M . This formula also shows that smoothness of the action Φ can be expressed in terms of smooth dependence of a flow on initial conditions.

We thus have seen that the smooth left-action of G on M is completely encoded in the foliation \mathcal{F} on $G \times M$ determined by the vector fields X^M , $X \in \mathfrak{g}$. In order to understand the obstructions that could prohibit the existence of such a left-action when a representation of \mathfrak{g} as vector fields on M is given, let us consider several simple examples, all variations on a single theme.

1.2. Example. Let $G = \mathbf{R}$ be the additive Lie group of real numbers with coordinate $g \in \mathbf{R}$, let $M = (0, 1) \subset \mathbf{R}$ be the open unit interval in \mathbf{R} with coordinate $x \in (0, 1)$. The Lie algebra \mathfrak{g} of G is (isomorphic to) \mathbf{R} and can be seen as the set of left-invariant vector fields $a \cdot \partial_g$, $a \in \mathbf{R}$ on $G = \mathbf{R}$. Associated to the Lie algebra element $a \cdot \partial_g$ we define the vector field $\rho(a \cdot \partial_g) = a \cdot \lambda \cdot \partial_x$ on M for some fixed constant $\lambda \in \mathbf{R}$. As the Lie algebra is 1-dimensional, this is automatically a representation.

The foliation \mathcal{F} on $G \times M$ is given by

$$\mathcal{F}|_{(g,x)} = \mathbf{R} \cdot (\partial_g + \lambda \partial_x)$$

and the leaf $L_{(g,m)}$ passing through (g, x) is given by

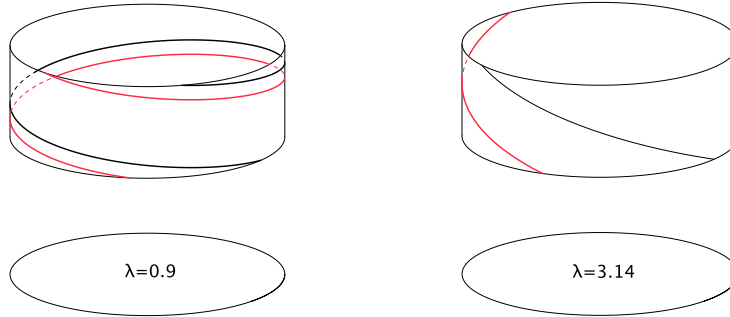
$$L_{(g,x)} = \{ (g + t, x + \lambda t) \mid -x < \lambda t < 1 - x \} .$$

In this case the projection $p : G \times M \rightarrow G$ restricted to a leaf is not a bijection (not surjective), unless $\lambda = 0$. Hence this representation ρ cannot be obtained by a left-action of G on M , unless $\lambda = 0$, in which case it is obtained by the trivial action $\Phi(g, x) = x$.

1.3. Example. If in example [1.2] we change the group G to the circle group $G = \mathbf{R}/\mathbf{Z}$, we can keep the rest unchanged. The formula defining \mathcal{F} is unchanged, but the leaf $L_{(g,x)}$ passing through (g, x) is given by

$$L_{(g,x)} = \{ (g + t + \mathbf{Z}, x + \lambda t) \mid -x < \lambda t < 1 - x \} .$$

As $G \times M$ is a (finite) cylinder, it is not hard to see that the leaves are parts of helices with slope λ . The projection $p : G \times M \rightarrow G$ restricted to a leaf will be surjective for $|\lambda| < 1$, but then it will not be injective unless $\lambda = 0$. On the other hand, for $|\lambda| \geq 1$ it will be injective but not surjective. Apart from the trivial case $\lambda = 0$ this representation of \mathfrak{g} thus never can be obtained from a left-action of G on M .



1.4. Example. If in example [1.3] we change the manifold M to $M = \mathbf{R}$, then the leaf $L_{(g,x)}$ passing through (g, x) is given by

$$L_{(g,x)} = \{ (g + t + \mathbf{Z}, x + \lambda t) \mid t \in \mathbf{R} \} .$$

As now $G \times M$ is an infinite cylinder, the projection $p : G \times M$ restricted to a leaf will always be surjective onto G , but (apart from the case $\lambda = 0$) it will never be injective.

1.5. Example. If in example [1.3] we change the manifold M to the circle $M = \mathbf{R}/\mathbf{Z}$, the formulæ still do not change, but then the leaf

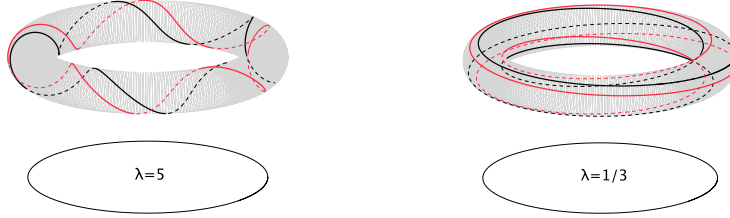
$L_{(g,x)}$ passing through (g, x) is given by

$$L_{(g,x)} = \{ (g + t + \mathbf{Z}, x + \lambda t + \mathbf{Z}) \mid t \in \mathbf{R} \} .$$

And now the topological nature of a leaf depends upon the value of $\lambda \in \mathbf{R}$: for $\lambda \in \mathbf{Q}$ it will be a circle and for $\lambda \in \mathbf{R} \setminus \mathbf{Q}$ it will be the real line. In all cases the projection $p : G \times M$ restricted to a leaf will be surjective onto G . However, for $\lambda \in \mathbf{R} \setminus \mathbf{Q}$ it will be (equivalent to) the standard universal covering map from \mathbf{R} to the circle \mathbf{R}/\mathbf{Z} . For $\lambda \in \mathbf{Q}$ it will be a finite covering map from the circle to itself, which is a bijection if and only if $\lambda \in \mathbf{Z}$. It follows that for $\lambda \notin \mathbf{Z}$ there does not exist a left-action of G on M producing the algebra representation as fundamental vector fields. On the other hand, it is easy to see that for $\lambda = n \in \mathbf{Z}$, the left-action $\Phi : G \times M \rightarrow M$ defined by

$$\Phi(g, x) = (x + ng)$$

induces the algebra representation as fundamental vector fields.



1.6. Example. If in examples [1.4] and [1.5] we change the group G to $G = \mathbf{R}$, then the projection $p : G \times M \rightarrow G$ will be a bijection for all $\lambda \in \mathbf{R}$. And it is easy to show that the representation of $\mathfrak{g} \cong \mathbf{R}$ as vector fields on M is induced by the left-action of $G = \mathbf{R}$ on M given by

$$(1.7) \quad \Phi(g, x) = x + \lambda g \text{ when } M = \mathbf{R} \quad \text{or} \quad \Phi(g, x) = x + \lambda g + \mathbf{Z} \text{ when } M = \mathbf{R}/\mathbf{Z}.$$

Before we analyse the examples in more detail, we first make the observation that the flow Φ_X of the fundamental vector field X^M on M is given by the formula

$$\Phi_X(t, m) = \Phi(\exp(-tX), m) ,$$

a fact we already used in formula (1.1). But this tells us that the flow of X^M is complete. It follows that a representation of \mathfrak{g} as vector fields on M can never be derived from a left-action when the vector fields $\rho(X)$ are not complete.

In view of this observation, it is immediate that in the examples [1.2] and [1.3] the algebra representation can not be derived from a left-action: on $M = (0, 1)$ the vector field $\lambda\partial_x$ is not complete (unless $\lambda = 0$ of course). On the other hand, on the examples [1.4], [1.5] and [1.6] the vector field $\lambda\partial_x$ always is complete. And we see that in these cases the restriction of the projection $p : G \times M \rightarrow G$ to a leaf of the foliation \mathcal{F} always is a covering map (which it is not in the first two examples, not even when it is surjective). In example [1.6] we used the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g} \cong \mathbf{R}$ and thus any covering from a connected set (and a leaf is by definition connected) must be a bijection.

We will show that these observations are general. More precisely, we will show that if we have a representation ρ of a finite dimensional Lie algebra \mathfrak{g} by smooth vector fields on a manifold M and if all vector fields $\rho(X)$ are complete, then the restriction of the projection $p : G \times M \rightarrow M$ to a leaf of the foliation \mathcal{F} is a covering map. It follows that if G is simply connected, then it must be a bijection. And hence we can define a left-action $\Phi : G \times M \rightarrow M$ by the condition

$$(1.8) \quad \Phi(g, m) = m' \quad \Longleftrightarrow \quad (e, m) \text{ and } (g, m') \text{ lie on the same leaf of the foliation } \mathcal{F}.$$

Smoothness of this action then can be deduced using formula (1.1). In this way we recover a variant of a result of R. Palais [Pa, Chapter IV, Thm III].

Theorem. *Let G be a simply connected Lie group with Lie algebra \mathfrak{g} and let ρ be a representation of \mathfrak{g} by smooth vector fields on a manifold M . If all vector fields $\rho(X)$ are complete, then there exists a unique smooth left-action $\Phi : G \times M \rightarrow M$ such that the representation ρ is given by the fundamental vector fields associated to this action.*

Our proof uses more or less the same ingredients as in [Pa], but we rely more heavily on the theory of covering maps. This allows us to replace the condition in [Pa] that the vector fields $\rho(X)$ should be proper by the slightly easier condition that they should be complete. A second reason to rely more heavily on topological arguments is that we wish to extend this result to the setting of representations of super Lie algebras by vector fields on (H^∞) -supermanifolds. In that context the natural (DeWitt) topology on a supermanifold is not separated/Hausdorff (but it is locally connected), and not all vector fields $\rho(X)$ will be smooth (only when X belongs to the body of the super Lie algebra). Moreover,

the individual maps $m \mapsto \Phi(g, m)$ for fixed $g \in G$ will in general not be smooth, even when Φ itself is smooth. This is a particular case of the more general situation that in super geometry families of maps can be smooth, whereas individual elements in the family will not be smooth. A similar problem occurs with foliations: not all leaves will have the structure of an immersed submanifold. These problems are easily solved by looking systematically at families of maps, instead of individual maps (this should not come as a surprise for people working with the sheaf-theoretic version of supermanifolds). A particularly useful example is the family of vector fields $\hat{X} = X^r - \rho(X)$ introduced in (1.1). Instead of looking at these vector fields one by one, one could look at the single vector field Z_A on $\mathfrak{g} \times G \times M$ defined by

$$Z|_{(X,g,m)} = 0|_X + X^r|_g - \rho(X)_m .$$

In standard differential geometry, if Z_A is a smooth vector field, then all vector fields \hat{X} will be smooth. In super differential geometry however, even if Z_A is smooth, not all individual vector fields \hat{X} will be smooth. That will be guaranteed only when X belongs to \mathbf{Bg} , the body of \mathfrak{g} .

As already hinted at above, we will work with the geometric H^∞ version of DeWitt supermanifolds, which is equivalent to the theory of graded manifolds of Leites and Kostant (see [DW], [Ko], [Le], [Ro] [Tu], [Va]). Any reader using a (slightly) different version of supermanifolds should be able to translate the results to her/his version of supermanifolds. The basic graded ring will be denoted as \mathcal{A} and we will think of it as the exterior algebra $\mathcal{A} = \Lambda V$ of an (unspecified) infinite dimensional real vector space V . Even (local) coordinates on a supermanifold belong to the even part \mathcal{A}_0 and odd coordinates belong to \mathcal{A}_1 . The body map \mathbf{B} projects any coordinate on its real part, the projection $\mathbf{B} : \mathcal{A} = \bigwedge V \rightarrow \bigwedge^0 V = \mathbf{R}$. The body map extends to any object in super differential geometry and projects onto a corresponding object in ordinary differential geometry. However, we will hardly need to distinguish between ordinary and super differential geometry, as the arguments are the same. Apart of course from the fact that we have to change our viewpoint to families instead of separate objects as explained above. On the other hand however, when we are only interested in topological properties, restricting attention to individual members of a family does not pose any problem. For instance, the map $\Phi_g : M \rightarrow M$ given by $\Phi_g(m) = \Phi(g, m)$ will always be a homeomorphism, even when it is not a diffeomorphism. This explains partly our insistence on topological arguments. As said, the argument applies as well to the ordinary differential geometric context as to the super

context. In the case of ordinary differential geometry, one can simplify the argument a little, using only differential geometric arguments, but those simplifications are minor indeed. They are outlined in [3.15].

Let us now give a brief outline of the argument. We start by recalling the definition of the leaf topology, which is finer than the original topology and for which the connected components are the leaves (in the super differential geometric context even the ones that are not immersed submanifolds). We then show that the flow Φ_A of the vector field Z_A introduced above is continuous with respect to the leaf topology (a direct consequence of Frobenius' theorem). This flow depends upon a time parameter t (which is even and thus belongs to \mathcal{A}_0), upon a Lie algebra element $X \in \mathfrak{g}$, an element in the group $g \in G$ and a point in the manifold $m \in M$. Fixing $t = 1$ (which we can as we assume that the flow is complete) and (g, m) , we use the fact that the exponential map is a diffeomorphism in a neighbourhood of the identity to create a map from a neighbourhood of any $g \in G$ to $G \times M$ via the dependence on X of the flow Φ_A . In the ordinary differential geometric context, the image is a local integral manifold for the foliation; in the super differential geometric context it is just continuous for the leaf topology. But being continuous for the leaf topology suffices to show that the canonical projection $p : G \times M \rightarrow M$ is a covering map for the leaf topology on $G \times M$. As we assume that G is simply connected, the restriction to any leaf is a homeomorphism. We thus can use the definition of the left-action in terms of leaves: $\Phi(g, m)$ is the unique point such that (e, m) and $(g, \Phi(g, m))$ lie on the same leaf. Smooth dependence on X of the flow Φ_A shows that the action defined this way is smooth in a neighbourhood of the identity and thus (because it is an action) it is globally smooth.

2. GENERALITIES ON FOLIATIONS

Definition. Let $f : M \rightarrow N$ be a map between two topological spaces. We will say that f is a *local homeomorphism* if for any $m \in M$ there exists an open neighbourhood $U \subset M$ of m and an open neighbourhood $V \subset N$ of $f(m)$ such that $f : U \rightarrow V$ is a homeomorphism.

Definition. Let $\mathcal{F} \subset TM$ be a foliation of rank k , i.e., an involutive subbundle of rank k . Let (x_1, \dots, x_n) be a local system of coordinates on the open subset $U \subset M$. We will say that the local coordinates (x_1, \dots, x_n) are adapted to the foliation if on U the foliation is spanned

by the tangent vectors $\partial_{x_1}, \dots, \partial_{x_k}$:

$$\forall m \in U \quad : \quad \mathcal{F}|_m = \left\{ \sum_{i=1}^k \alpha_i \cdot \partial_{x_i}|_m \mid \alpha_i \in \mathcal{A} \right\}.$$

By Frobenius' theorem, around every point there exists local coordinates adapted to the foliation. Now let (x_1, \dots, x_n) be a local coordinate system on $U \subset M$ adapted to the foliation and choose $a_i \in \mathcal{A}_{\varepsilon_i}$, $i = k+1, \dots, n$ (where ε_i indicates the parity of the coordinate x_i). We then can define the slice $U_{a_{>k}}$ by

$$U_{a_{>k}} = \{ m \in U \mid x_{k+1}(m) = a_{k+1}, \dots, x_n(m) = a_n \}.$$

Trivial Lemma. *Let (x_1, \dots, x_n) be local coordinates on $U \subset M$, let $a_i \in \mathcal{A}_{\varepsilon_i}$ be arbitrary and let $V \subset M$ be any open subset. Then (x_1, \dots, x_n) are local coordinates on $U \cap V$ and we have the equality*

$$(U \cap V)_{a_{>k}} = U_{a_{>k}} \cap V.$$

2.1. Lemma. *The collection \mathcal{B} of all connected components of all slices of all local coordinate systems adapted to the foliation forms the basis of a topology for M , a topology that is finer than the original topology of M .*

Proof. To prove that \mathcal{B} is the basis for a topology, we have to prove the property

$$\forall B_1, B_2 \in \mathcal{B} \quad \forall m \in B_1 \cap B_2 \quad \exists B_3 \in \mathcal{B} : m \in B_3 \subset B_1 \cap B_2.$$

So suppose $B_1, B_2 \in \mathcal{B}$ and $m \in B_1 \cap B_2$. By definition there exist local coordinate systems (x_1, \dots, x_n) on an open $U \subset M$, local coordinates (y_1, \dots, y_n) on an open $V \subset M$ both adapted to the foliation, and $a_i, b_i \in \mathcal{A}_{\varepsilon_i}$, $i > k$, such that:

B_1 a connected component of $U_{a_{>k}}$ and

B_2 a connected component of $V_{b_{>k}}$.

We claim that B_3 defined as the connected component of $U_{a_{>k}} \cap V = (U \cap V)_{a_{>k}}$ containing m satisfies the requirement. It obviously belongs to \mathcal{B} and we obviously have the inclusion $B_3 \subset B_1$ as it is connected and included in $U_{a_{>k}}$. To prove that it is included in $V_{b_{>k}}$, we note that, by definition of coordinate systems, there exist smooth maps ψ_i on $U \cap V$ such that $y_i = \psi_i(x_1, \dots, x_n)$ on $U \cap V$. The fact that both

coordinate systems are adapted to the foliation implies that we must have

$$(2.2) \quad \left(\frac{\partial \psi_j}{\partial x_i}(m) \right)_{i,j=1}^k \text{ an invertible matrix} \quad \text{and} \quad \frac{\partial \psi_j}{\partial x_i}(m) = 0 \text{ for } i \leq k \text{ and } j > k.$$

In particular, the functions $y_j = \psi_j(x_1, \dots, x_n)$ with $j > k$ are constant on B_3 as it is connected and contained in $U_{a_{>k}}$. But because $m \in V_{b_{>k}}$, we must have $y_j(m) = b_j$ for $j > k$, and thus $B_3 \subset V_{b_{>k}}$. As B_3 is connected and contains m , we thus must have the inclusion $B_3 \subset B_2$ as wanted.

To show that the associated topology is finer than the original topology, it suffices to make two remarks. First that we have the obvious equality

$$U = \bigcup_{a_{k+1}, \dots, a_n} U_{a_{k+1}, \dots, a_n}.$$

And second that we can take arbitrary small coordinate systems adapted to the foliation, simply by taking intersections with arbitrary open subsets (of the original topology). \square

2.3. Definition. The topology on M defined by the basis given in [2.1] is called the *leaf topology* of M and denoted as $\mathcal{T}_{\mathcal{F}}$. In non-graded (non-super) geometry, the connected components of M with respect to this topology are exactly the leaves of the foliation, i.e., the immersed maximal integral manifolds of the foliation. In super geometry some (most) of the connected components do not have the structure of an immersed submanifold, and thus are not leaves according to the official definition. That is why, in the super differential geometric context, we “have to” use the leaf topology: the collection of all immersed maximal integral manifolds do not fill up the manifold M . Via the connected components for the leaf topology we have acces to “all” leaves. But then we have to forego differential geometric arguments, as these connected components do not in general have the structure of an immersed supermanifold.

2.4. Lemma. *Let (x_1, \dots, x_n) be a local system of coordinates on the open set U adapted to the foliation. Then the topology on a slice $U_{a_{>k}}$ induced by the leaf topology is the same as the the one induced by the original topology on M .*

Proof. As the leaf topology is finer than the original topology, it follows immediately that the topology induced on $U_{a_{>k}}$ by the original topology is included in the topology induced on $U_{a_{>k}}$ by the leaf topology.

For the converse, choose $m \in U_{a_{>k}}$ and a basic open neighbourhood B_2 of m for the leaf topology, i.e., B_2 is the connected component of some slice $V_{b_{>k}}$. In the proof of [2.1] we have shown that B_3 , the connected component of $(U \cap V)_{a_{>k}}$ containing m , is included in the intersection $U_{a_{>k}} \cap B_2$. As (x_1, \dots, x_n) are local coordinates on $U \cap V$, it follows that there exists an open set $m \in W \subset U \cap V$ of the form $W = W_1 \times W_2$ with (x_1, \dots, x_k) local coordinates on W_1 and (x_{k+1}, \dots, x_n) local coordinates on W_2 . Moreover, we may assume that W_1 is connected. We thus have:

$$W_1 \times \{(a_{k+1}, \dots, a_n)\} = W_{a_{>k}} \subset (U \times V)_{a_{>k}} .$$

As m belongs to the connected set $W_1 \times \{(a_{k+1}, \dots, a_n)\} = W_{a_{>k}}$, it follows that we have the inclusions

$$W \cap U_{a_{>k}} = W_{a_{>k}} \subset B_3 \subset U_{a_{>k}} \cap B_2 .$$

This shows that any open neighbourhood in $U_{a_{>k}}$ of m for the topology induced by the leaf topology contains an open neighbourhood in $U_{a_{>k}}$ of m for the topology induced by the original topology of M . And hence the topology on $U_{a_{>k}}$ induced by the original topology is finer than the one induced by the leaf topology. \boxed{QED}

Remark. While it is true that the topology on a slice $U_{a_{>k}}$ is the same whether induced by the original topology or by the leaf topology, it is **not** true that the topology on a connected component of the leaf topology is induced by the original topology. It suffices to think of the torus $(\mathbf{R}/\mathbf{Z})^2$ foliated by lines with an irrational slope. The connected components are the leaves of this foliation, which are homeomorphic to the real line. But as these lines are dense in the torus, the topology induced on such a leaf by the topology of the torus is not the same as the topology of the real line: no finite interval on the line can be the intersection of an open set in the torus with a leaf.

3. REPRESENTATIONS OF A (SUPER) LIE ALGEBRA AS VECTOR FIELDS

Definition. Let \mathfrak{g} be a (super) Lie algebra of dimension d , let e_1, \dots, e_d be a (homogeneous) basis and let $c_{ij}^k \in \mathbf{R}$ be the associated structure constants: $[e_i, e_j] = \sum_{k=1}^d c_{ij}^k e_k$. An even (left-)linear map ρ

from \mathfrak{g} to the space of sections of the tangent bundle TM of a (super) manifold M is called a *smooth representation of \mathfrak{g} on M* if the vector fields $\rho(e_i)$ are smooth (necessarily of the same parity as e_i because ρ is assumed to be even) and satisfy the commutation relations $[\rho(e_i), \rho(e_j)] = \sum_{k=1}^d c_{ij}^k \rho(e_k)$. An equivalent condition would be to require that $\rho(X)$ is smooth for all $X \in \mathbf{Bg}$ (the $X \in \mathfrak{g}$ having real coordinates) and such that $[\rho(X), \rho(Y)] = \rho([X, Y])$ for all $X, Y \in \mathbf{Bg}$.

Remarks. A (left-)linear map ρ from \mathfrak{g} to the space of sections of the tangent bundle TM such that the vector fields $\rho(e_i)$ are smooth of the same parity as e_i can be seen as a single even smooth vector field Z on $\mathfrak{g}_0 \times M$ defined as

$$Z|_{(X,m)} = 0|_X + \rho(X)|_m = 0|_X + \sum_{i=1}^d X^i \cdot \rho(e_i)|_m ,$$

where $X = \sum_{i=1}^d X^i \cdot e_i$ is the decomposition of $X \in \mathfrak{g}$ with respect to the given basis. Since the $\rho(e_i)$ are smooth, the last equality immediately shows that Z is smooth on the product $\mathfrak{g}_0 \times M$. With a slight abuse of notation, one could say that we have $Z|_{(X,m)} = \rho(X)|_m$: Z at (X, m) equals $\rho(X)$ at m .

We can extend the idea of the global vector field Z to encode the representation property. To do so, we consider the two global smooth vector fields Z_1 and Z_2 on $\mathfrak{g} \times \mathfrak{g} \times M$ defined by

$$Z_1|_{(X,Y,m)} = 0|_X + 0|_Y + \rho(X)|_m \quad , \quad Z_2|_{(X,Y,m)} = 0|_X + 0|_Y + \rho(Y)|_m .$$

With the same abuse of notation as above, we thus can write $Z_1|_{(X,Y,m)} = \rho(X)|_m$ and $Z_2|_{(X,Y,m)} = \rho(Y)|_m$. The representation condition can then be stated as saying that we should have the equality $[Z_1, Z_2]|_{(X,Y,m)} = \rho([X, Y])|_m$. The passage via the smooth vector fields Z_1 and Z_2 avoids the problem that the individual vector fields $\rho(X)$ need not be smooth, and thus that their commutator is not defined.

3.1. Definition. Let G be a (super) Lie group with (super) Lie algebra \mathfrak{g} and let ρ be a smooth representation of \mathfrak{g} on a (super) manifold M of total dimension n . Associated to these data, we define the distribution (subbundle) $\mathcal{F} \subset T(G \times M)$ as the subbundle spanned by the smooth vector fields $e_i^r - \rho(e_i)$, where e_i^r denotes the right-invariant vector field on G whose value at the identity $e \in G$ is $e_i \in \mathfrak{g} \cong T_e G$.

More precisely:

$$\begin{aligned}\mathcal{F}|_{(g,m)} &= \left\{ \sum_{i=1}^d X_i \cdot (e_i^r|_g - \rho(e_i)|_m) \mid X_i \in \mathcal{A} \right\} \\ &= \left\{ X^r|_g - \rho(X)|_m \mid X \in \mathfrak{g} \right\},\end{aligned}$$

where X^r denotes the right-invariant vector field on G whose value at $e \in G$ is $X \in \mathfrak{g} \cong T_e G$.

We denote by $p : G \times M \rightarrow G$ the canonical projection $p(g, m) = g$. It then is immediate that the tangent map $T_{(g,m)} p : T_{(g,m)}(G \times M) \rightarrow T_g G$ is an isomorphism from $\mathcal{F}|_{(g,m)}$ to $T_g G$.

Lemma. *The distribution $\mathcal{F} \subset T(G \times M)$ is involutive, i.e., \mathcal{F} is a foliation on $G \times M$.*

Proof. For the right-invariant vector fields e_i^r on G we have the equalities $[e_i^r, e_j^r] = -\sum_{k=1}^d c_{ij}^k e_k^r$. It then follows immediately from the fact that ρ is a representation and the use of the minus sign in the definition of \mathcal{F} that it is involutive. \square QED

Terminology. We will talk about “leaf-open” subsets and “leaf-continuous” maps whenever we equip $G \times M$ with the leaf topology $\mathcal{T}_{\mathcal{F}}$ [2.3] associated to the foliation \mathcal{F} defined in [3.1] and when this space appears alone or in a direct product.

3.2. Proposition. *The map $p : G \times M \rightarrow G$ is a local leaf-homeomorphism, and thus in particular leaf-continuous.*

Proof. What we have to show is that for all $(g, m) \in G \times M$ there exists a leaf-open neighbourhood O of (g, m) and an open neighbourhood V of g such that $p : O \rightarrow V$ is a homeomorphism (when O is equipped with the topology induced by the leaf topology). So choose $(g, m) \in G \times M$, let U be an open neighbourhood of (g, m) with local coordinates x_1, \dots, x_{d+n} adapted to the foliation \mathcal{F} and let y_1, \dots, y_d be local coordinates in an open neighbourhood of $g \in G$. As the foliation is spanned by the tangent vectors ∂_{x_i} , $i \leq d$ and as the tangent map of the projection $p : G \times M \rightarrow G$ is an isomorphism from $\mathcal{F}|_{(g,m)}$ to $T_g G$, it follows that the $d \times d$ matrix

$$\left(\frac{\partial y_i}{\partial x_j}(g, m) \right)_{i,j=1}^d$$

is invertible. By the inverse function theorem it follows that there exists an open neighbourhood $U' \subset U$ of (g, m) such that $y_1, \dots, y_d, x_{d+1}, \dots, x_{d+n}$ forms a local system of coordinates (on $U' \subset G \times M$) adapted to the foliation. Without loss of generality we may assume that U' is of the form $V \times W$ with (y_1, \dots, y_d) local coordinates on V , which thus is an open neighbourhood of $g \in G$, and (x_{d+1}, \dots, x_n) local coordinates on W . Moreover, the (original) topology of $U' = V \times W$ is the product topology. It follows that the topology induced on the slice $U'_{a_{>d}} = V \times \{(a_{d+1}, \dots, a_{d+n})\}$ “is” the topology of V , and thus the projection $p : U'_{a_{>d}} \rightarrow V$ is a homeomorphism. But $O = U'_{a_{>d}}$ is open in the leaf topology and the topology on $O = U'_{a_{>d}}$ induced by the leaf topology is the same as the one induced by the original topology [2.4], and thus p is a local leaf-homeomorphism. $\square QED$

Definition. Associated to a smooth representation of \mathfrak{g} on M we define three even smooth vector fields Z_R, Z_M and Z_A on $\mathfrak{g}_0 \times G, \mathfrak{g}_0 \times M$ and $\mathfrak{g}_0 \times G \times M$ respectively by

$$\begin{aligned} Z_R|_{(X,g)} &= 0|_X + X^r|_g \quad , \quad Z_M|_{(X,m)} = 0|_X + \rho(X)|_m \\ \text{and} \quad Z_A|_{(X,g,m)} &= 0|_X + X^r|_g - \rho(X)|_m \quad , \end{aligned}$$

where as before X^r denotes the right-invariant vector field on G whose value at $e \in G$ is X . We denote by Φ_R, Φ_M and Φ_A their respective flows. It is immediate that Φ_R is defined on the whole of $\mathcal{A}_0 \times \mathfrak{g}_0 \times G$ (Z_R is complete) and is given in terms of the exponential map by

$$\Phi_R(t, X, g) = (X, \exp(tX)g) \quad .$$

(Nota Bene: it might be better to say that the exponential map is defined using the flow Z_A by this equation, but that is of less importance here.) On the other hand, the flow Φ_M is defined on an open subset $W_M \subset \mathcal{A}_0 \times \mathfrak{g}_0 \times M$ satisfying the condition that for each $(X, m) \in \mathfrak{g}_0 \times M$ the set

$$I_{(X,m)} = \{t \in \mathcal{A}_0 \mid (t, X, m) \in W_M\} = \mathcal{A}_0 \times \{(X, m)\} \cap W_M \subset \mathcal{A}_0$$

is connected and contains 0 (the domain of definition of the maximal integral curve passing at $t = 0$ through (X, m) , but in super differential geometry not all these separate curves are differentiable).

3.3. Lemma. *There exists a smooth function $\Psi_M : W_M \rightarrow M$ such that the flow Φ_M is given by*

$$\Phi_M(t, X, m) = (X, \Psi_M(t, X, m)) \quad .$$

Moreover, the domain of definition $W_A \subset \mathcal{A}_0 \times \mathfrak{g}_0 \times G \times M$ of the flow Φ_A is given by

$$W_A = \{ (t, X, g, m) \mid (t, X, m) \in W_M \}$$

and its flow Φ_A by

$$\Phi_A(t, X, g, m) = (X, \exp(tX)g, \Psi_M(t, X, m)) .$$

Proof. Since Φ_M is smooth, the composition with the projections onto either \mathfrak{g}_0 of M is smooth. As the vector field Z_M is zero in the direction of X , the X -component of Φ_M must be constant, proving the first part.

If we denote by p_{12} and p_{13} the projections $p_{12} : \mathfrak{g}_0 \times G \times M \rightarrow \mathfrak{g}_0 \times G$, $(X, g, m) \mapsto (X, g)$ and $p_{13} : \mathfrak{g}_0 \times G \times M \rightarrow \mathfrak{g}_0 \times M$, $(X, g, m) \mapsto (X, m)$, then it is immediate that their tangent maps intertwine the vector fields:

$$Tp_{12}(Z_A|_{(X,g,m)}) = Z_G|_{(X,g)} \quad \text{and} \quad Tp_{13}(Z_A|_{(X,g,m)}) = Z_M|_{(X,m)} .$$

It follows that these maps intertwine their flows:

$$\begin{aligned} p_{12}(\Phi_A(t, X, g, m)) &= \Phi_R(t, p_{12}(X, g, m)) = \Phi_R(t, X, g) \\ &= (X, \exp(tX)g) \end{aligned}$$

and

$$\begin{aligned} p_{13}(\Phi_A(t, X, g, m)) &= \Phi_M(t, p_{13}(X, g, m)) \\ &= \Phi_M(t, X, m) = (X, \Psi_M(t, X, m)) . \end{aligned}$$

As the flow Φ_A necessarily is of the form

$$\Phi_A(t, X, g, m) = (\Phi_{A,X}(t, X, g, m), \Phi_{A,g}(t, X, g, m), \Phi_{A,m}(t, X, g, m))$$

for smooth functions with values in \mathfrak{g}_0 , G and M respectively, the result on Φ_A follows immediately. But it also shows that Φ_A is defined at least on the given expression for W_A . And as the flow can not be defined for values of $t \in \mathcal{A}_0$ for which the projection is not defined, W_A must be given by this expression. \square

3.4. Remark. If we look at a slice $\{X\} \times M$, then the restriction of the vector field Z_M is tangent to this slice and equals the vector field $\rho(X)$. It follows that we can interpret the map $(t, m) \mapsto \Phi_M(t, X, m)$ as the flow of this vector field. In the non-graded case this means that we can group the flows of all these vector fields together to form the flow of Z_M , the only bonus of looking at Z_M being that we automatically have a smooth dependence on X . On the other hand, in the super case, not all slices are genuine submanifolds and not all $\rho(X)$ are smooth vector

fields. And thus in the super case, the passage via Z_M is obligatory. Analogous remarks hold for the vector fields Z_A and Z_R .

3.5. Lemma. *The vector field Z_A on $\mathfrak{g}_0 \times G \times M$ is complete if and only if the vector field Z_M on $\mathfrak{g}_0 \times M$ is complete, which is the case if and only if all vector fields $\rho(X)$ on M with $X \in \mathbf{B}\mathfrak{g}_0$ are complete.*

Proof. That Z_A is complete if and only if Z_M is complete is a direct consequence of [3.3]. On the other hand, the domain of definition of the flow of a smooth vector field on a super manifold is completely determined by the domain of definition of the flow of the body of the vector field on the body of the supermanifold. As we have

$$\mathbf{B}Z_M|_{(X,m)} = 0|_{\mathbf{B}X} + \rho(\mathbf{B}X)|_{\mathbf{B}m} ,$$

and as these objects live in ordinary differential geometry, we can apply the argument of [3.4] and conclude that the flow of $\mathbf{B}Z_M$ consists of the family of flows for the separate (smooth!) vector fields $\rho(\mathbf{B}X)$. It follows that Z_M is complete if and only if all vector fields $\rho(X)$, $X \in \mathbf{B}\mathfrak{g}_0$ are complete on $\mathbf{B}M$. Reversing the argument on the domain of a flow, this will be the case if and only if all (smooth) vector fields $\rho(X)$, $X \in \mathfrak{g}_0$ are complete on M . \square *QED*

3.6. Proposition. *The flow Φ_A is leaf-continuous.*

Proof. What we have to show is the following. Suppose we are given a point $(t_o, X_o, g_o, m_o) \in W_A \subset \mathcal{A}_0 \times \mathfrak{g}_0 \times G \times M$, an open set $W_1 \subset \mathfrak{g}_0$ and a basic open set $V_{b>d}^c \subset G \times M$ for the leaf topology (where the superscript c here and in the sequel indicates that we take the appropriate connected component) such that $\Phi_A(t_o, X_o, g_o, m_o) \in W_1 \times V_{b>d}^c$. Then we have to find open neighbourhoods $I_0 \subset \mathcal{A}_0$ of t_o , $W_0 \subset \mathfrak{g}_0$ of X_o and $(U_0)_{a>d}^c \subset G \times M$ (a basic open set for the leaf topology) of (g_o, m_o) such that we have the inclusion

$$\Phi_A(I_0 \times W_0 \times (U_0)_{a>d}^c) \subset W_1 \times V_{b>d}^c .$$

To prove this, we proceed in two steps. In the first step we show that it is true for all points in W_A with t “sufficiently small.” And in the second step we show that it is true for all points in W_A .

For the first step, we start with a local coordinate system x_1, \dots, x_{d+n} on an open set $U \subset G \times M$ adapted to the foliation. It follows immediately that the vector field Z_A has the local expression on $\mathfrak{g}_0 \times U$

$$Z_A|_{(X_i, x_j)} = \sum_{i,j=1}^d X_i \cdot f_{ij}(x) \cdot \frac{\partial}{\partial x_j}|_{(X_i, x_j)} ,$$

where X_i denote the coordinates of $X \in \mathfrak{g}_0$ with respect to the fixed basis e_1, \dots, e_n of \mathfrak{g} : $X = \sum_{i=1}^d X_i e_i$, and where f_{ij} are smooth functions on $G \times M$ (for fixed i they represent the coefficients of the vector fields $e_i^r - \rho(e_i)$ on $G \times M$ with respect to the coordinate system x). Local existence and uniqueness of ordinary differential equations then tells us that for any $(X_o, g_o, m_o) \in \mathfrak{g}_0 \times U$ there exists a connected open neighbourhood $I_o \subset \mathcal{A}_0$ of 0, a connected open neighbourhood $W_o \subset \mathfrak{g}_0$ of X_o and an open neighbourhood $U_o \subset U$ of (g_o, m_o) such that there exists a (unique) local flow

$$\Phi_A : I_o \times W_o \times U_o \rightarrow \mathfrak{g}_0 \times U$$

for the restriction of the vector field Z_A to $\mathfrak{g}_0 \times U$. As the coordinates X_i and x_j can be used as well in the source as in the target space of this local flow Φ_A , and because the components of Z_A in the direction of ∂_{X_i} and ∂_{x_j} for $j > d$ are zero, we can write this local flow sloppily as

$$\Phi_A(t, X_i, x_j) = (X_i, x_j(t)) \quad \text{with} \quad x_j(t) = x_j \text{ for } j > d.$$

Now let $(t, X, g, m) \in I_o \times W_o \times U_o$ be arbitrary and let $W_1 \times V_{b>d}^c$ be a basic open neighbourhood of $\Phi_A(t, X, g, m)$ for the leaf topology. As $\Phi_A(t, X, g, m)$ also belongs to $\mathfrak{g}_0 \times U$, there exists a_{d+1}, \dots, a_{d+n} such that $\Phi_A(t, X, g, m) \in W_1 \times U_{a>d}$. According to the proof of [2.1], we thus have

$$\Phi_A(t, X, g, m) \in W_1 \times (V \cap U)_{a>d}^c \subset W_1 \times V_{b>d}^c .$$

As Φ_A is continuous for the usual topologies, there exist connected open neighbourhoods $I'_o \subset I_o$ of t , $W'_o \subset W_o$ of X and $U'_o \subset U_o$ such that we have

$$\Phi_A(I'_o \times W'_o \times U'_o) \subset W_1 \times (U \cap V) .$$

As $x_j(t) = x_j$ for $j > d$ we have (using connectedness!)

$$\Phi_A(I'_o \times W'_o \times (U'_o)_{a>d}^c) \subset W_1 \times (U \cap V)_{a>d}^c \subset W_1 \times V_{b>d}^c ,$$

which shows that Φ_A is continuous for the leaf topology at all points in $I_o \times W_o \times U'$. We thus have shown that for all points $(X_o, g_o, m_o) \in \mathfrak{g}_0 \times G \times M$ there exists a basic open neighbourhood $I_o \times W_o \times U'$ of

$(0, X_o, g_o, m_o) \in W_A$ such that Φ_A is leaf-continuous for all points in this neighbourhood.

For the second step we fix (X, g, m) and define $C \subset \mathcal{A}_0$ as

$$(3.7) \quad C = \{ t \in \mathcal{A}_0 \mid (t, X, g, m) \in W_A \text{ and } \Phi_A \text{ leaf-continuous at } (t, X, g, m) \} .$$

We then note that according to the first step, C contains an open interval containing 0. So let $t \in \overline{C}$ such that $(t, X, g, m) \in W_A$. We now apply the first step to the point $\Phi_A(t, X, g, m)$ and conclude that there exist open subsets $I_o \subset \mathcal{A}_0$ containing 0, $W_o \subset \mathfrak{g}_0$ and $U_o \subset G \times M$ such that $\Phi_A(t, X, g, m) \in W_o \times U_o$ and such that Φ_A is leaf-continuous at all points in $I_o \times W_o \times U_o \subset W_A$.

As Φ_A is continuous, $\Phi_A^{-1}(W_o \times U_o) \subset W_A$ is open, and thus there exists an open interval $I_2 \subset \mathcal{A}_0$ containing t such that $I_2 \times \{(X, g, m)\} \subset W_A$. As I_o is open and contains 0, we may assume (by taking a smaller I_2 if needed) that we also have the inclusion

$$I_2 - I_2 \equiv \{ s - s' \mid s, s' \in I_2 \} \subset I_o .$$

Now take $t'' \in I_2$ arbitrary and note that, because t is in the closure of C , there exists $t' \in C \cap I_2$. And thus in particular Φ_A is leaf-continuous at (t', X, g, m) . But the group law of a flow tells us that we have the equality

$$\Phi_A(t'', X, g, m) = \Phi_A(t'' - t', \Phi_A(t', X, g, m)) .$$

Moreover, we have $\Phi_A(t', X, g, m) \in W_o \times U'$ and $t - t' \in I_2 - I_2 \subset I_o$ and thus Φ_A is leaf-continuous at $(t'' - t', \Phi_A(t', X, g, m))$. And thus by composition of maps, Φ_A is leaf-continuous at (t'', X, g, m) . It follows that C is open and closed in $\mathcal{A}_0 \times \{(X, g, m)\} \cap W_A$, which is connected. And thus we have the equality $C = \mathcal{A}_0 \times \{(X, g, m)\} \cap W_A$. The final conclusion is that Φ_A is leaf-continuous at all points of W_A . \square

3.8. Proposition. *If the flow of all smooth vector fields $\rho(X)$, $X \in \mathbf{Bg}_0$ is complete, then the map $p : G \times M \rightarrow G$ is a covering map when we equip $G \times M$ with the leaf topology*

Proof. Let $g \in G$ be fixed, then $p^{-1}(\{g\}) = \{g\} \times M$. We will show that there exist an open neighbourhood V of g and leaf-open sets $U_m \subset G \times M$ for each $m \in M$ such that $p^{-1}(V) = \cup_{m \in M} U_m$, $m \neq m' \Rightarrow U_m \cap U_{m'} = \emptyset$ and such that $p : U_m \rightarrow V$ is a leaf-homeomorphism. And these properties are exactly the condition for p to be a covering map (for the leaf topology).

Let $O \subset \mathfrak{g}_0$ and $P \subset G$ be open neighbourhoods of $0 \in \mathfrak{g}_0$ and the identity $e \in G$ respectively such that the exponential map $\exp : \mathfrak{g}_0 \rightarrow G$ is a diffeomorphism from O to P . It follows that $V = P \cdot g = R_g(P)$ (the right-translate of P over $g \in G$) is an open neighbourhood of g .

As all vector fields $\rho(X)$, $X \in \mathbf{Bg}$ are complete, the vector field Z_A is complete [3.5], which means in particular that $\Phi_A(1, X, g, m)$ is defined for all (X, g, m) . We thus can define, for any $m \in M$, the composite map $\psi_m : V \rightarrow G \times M$ by

$$\psi_m(h) = (p_{23} \circ \Phi_A)(1, \exp^{-1}(hg^{-1}), g, m) ,$$

where $p_{23} : \mathfrak{g}_0 \times G \times M \rightarrow G \times M$ denotes the projection $p_{23}(X, g, m) = (g, m)$, which is certainly leaf-continuous. As a right-translation (here $R_{g^{-1}}$) is a homeomorphism (for non-super manifolds it always is a diffeomorphism, for super manifolds it is a diffeomorphism when g has real coordinates, i.e., is in the body of G), as \exp^{-1} is a diffeomorphism on P hence a homeomorphism and as Φ_A is leaf-continuous and thus leaf-continuous when applied to a subset (here the fixed values of $t = 1 \in \mathcal{A}_0$ and $(g, m) \in G \times M$), the map ψ_m is leaf-continuous. But according to [3.3] we also have the equality

$$(p_2 \circ \Phi_A)(1, \exp^{-1}(hg^{-1}), g, m) = \exp(1 \cdot \exp^{-1}(hg^{-1}))g = h ,$$

where $p_2 : \mathfrak{g}_0 \times G \times M \rightarrow G$ denotes the projection on the second factor. It follows immediately that we have the equality

$$p \circ \psi_m = id|_V .$$

As p and ψ_m are leaf-continuous [3.2], [3.6], it follows that $p : \psi_m(V) \rightarrow V$ is a leaf-homeomorphism with ψ_m as its inverse.

Now choose $x \in \psi_m(V)$ and define $h = p(x)$. Since p is locally a leaf-homeomorphism [3.2], there exists a leaf-open neighbourhood U' of x and an open neighbourhood $V' \subset V$ of h such that $p : U' \rightarrow V'$ is a leaf-homeomorphism. Since ψ_m is leaf-continuous, $V'' = V' \cap \psi_m^{-1}(U') \subset V'$ is an open neighbourhood of h . Now $\psi_m(V'') \subset U' \cap \psi_m(V)$ and on $\psi_m(V)$ the map ψ_m is the inverse of p . But p is a leaf-homeomorphism on U' and thus $\psi_m(V'') = p^{-1}(V'')$ is leaf-open, which shows that x has a leaf-open neighbourhood contained in $\psi_m(V)$. This shows that $\psi_m(V)$ is leaf-open.

To finish the proof, it thus remains to show that we have $p^{-1}(V) = \bigcup_{m \in M} \psi_m(V)$ and the implication $m \neq m' \Rightarrow \psi_m(V) \cap \psi_{m'}(V) = \emptyset$. Both these properties follow from the group property of a flow. For the first, pick $x \in p^{-1}(V)$ and define $h = p(v) \in V$. Hence there exists $X \in O$ such that $h = \exp(X)g$, i.e., $X = \exp^{-1}(hg^{-1})$. With this X

we compute, using [3.3]

$$p_2(\Phi_A(-1, X, x)) = \exp(-1 \cdot X)h = (hg^{-1})^{-1}h = g .$$

It follows that there exists $m \in M$ such that

$$\Phi_A(-1, X, x) = (X, g, m) .$$

But then we can use the group property to compute:

$$\begin{aligned} (X, x) &= \Phi_A(1, \Phi_A(-1, X, x)) = \Phi_A(1, X, g, m) \\ &= \Phi_A(1, \exp^{-1}(hg^{-1}), g, m) , \end{aligned}$$

from which it follows immediately that we have $x = \psi_m(h)$. Hence we have the equality $p^{-1}(V) = \cup_{m \in M} \psi_m(V)$.

For the last property, suppose $x \in \psi_m(V) \cap \psi_{m'}(V)$ and define $h = p(x)$. By definition of ψ_m and $\psi_{m'}$, the equality $\psi_m(h) = x = \psi_{m'}(h)$ implies that we (also) have the equality

$$\Phi_A(1, \exp^{-1}(hg^{-1}), g, m) = \Phi_A(1, \exp^{-1}(hg^{-1}), g, m') .$$

Applying $\Phi_A(-1, \cdot)$ to both sides (and using the group property of a flow) then tells us that we have the equality

$$(\exp^{-1}(hg^{-1}), g, m) = (\exp^{-1}(hg^{-1}), g, m') ,$$

and thus in particular $m = m'$. \square

3.9. Lemma. *The map $R : (G \times M) \times G \rightarrow G \times M$ defined as $R((g, m), h) = (gh, m)$ is leaf-continuous. In particular for fixed $h \in G$ and $L \subset G \times M$ a leaf-connected component (for the leaf topology on $G \times M$), the set $R_h(L) \subset G \times M$ is also a leaf-connected component, where $R_h : G \times M \rightarrow G \times M$ is the map $(g, m) \mapsto (gh, m)$.*

Proof. Choose $((g, m), h) \in (G \times M) \times G$ and $V \subset G \times M$ an open neighbourhood of (gh, m) with local coordinates y_1, \dots, y_{d+n} adapted to the foliation. It follows that there exist b_{d+1}, \dots, b_{d+n} such that $V_{b_{>d}}^c$ is a leaf-open neighbourhood of (gh, m) (and as before, the superscript c indicates that we take the appropriate connected component). As R is smooth, it is in particular continuous, so there exist open neighbourhoods $U \subset G \times M$ and $W \subset G$ of (g, m) and h respectively such that $R(U \times W) \subset V$. By taking a smaller U if necessary, we may assume without loss of generality that there exists local coordinates x_1, \dots, x_{d+n} on U adapted to the foliation. By taking a smaller W if necessary, we may also assume that W is connected.

Now consider the vector field X_P on $(G \times M) \times G$ defined as

$$X_P|_{((g,m),h)} = e_i^r|_g - \rho(e_i)|_m + 0|_h ,$$

where e_i is one of the basis elements of the (super) Lie algebra \mathfrak{g} . Then, as e_i^r is right-invariant, it follows immediately that the tangent map of R produces the image

$$TR(X_P|_{((g,m),h)}) = e_i^r|_{gh} - \rho(e_i)|_m .$$

With a slight abuse of notation, this means that TR maps the foliation \mathcal{F} (extended to the product $(G \times M) \times G$) to itself. If we now denote by z_i local coordinates on $W \subset G$, then we can write (for any $1 \leq i \leq n + d$):

$$y_i = R_i(x, z) .$$

But on $U \times W$ the foliation is spanned by ∂_{x_i} for $i \leq d$ and on V it is spanned by ∂_{y_i} for $i \leq d$. The fact that R maps the foliation to itself thus implies that we must have

$$\frac{\partial R_i}{\partial x_j}(x, z) = 0 \quad \text{for } i > d \text{ and } j \leq d .$$

And thus y_i , $i > s$ is constant on $U_{a>d}^c \times W$, which implies that we have the inclusion

$$R(U_{a>d}^c \times W) \subset V_{b>d}^c$$

as wanted. $\square QED$

3.10. Lemma. *Let G be a connected (super) Lie group, M a (super) manifold and $\Phi : G \times M \rightarrow M$ a (set theoretic) left-action of G on M . If there exists an open neighbourhood U of the identity element $e \in G$ such that the restriction $\Phi : U \times M \rightarrow M$ is smooth, then Φ is globally smooth, i.e., Φ is a smooth left-action of G on M .*

Proof. As G is connected, any open neighbourhood U of the identity generates G , i.e., any $g \in G$ is the product of a finite number of elements of U . But if we have

$$g = h_1 \cdot h_2 \cdots h_n ,$$

with $h_i \in U$, then we also have

$$\mathbf{B}g = (\mathbf{B}h_1) \cdot (\mathbf{B}h_2) \cdots (\mathbf{B}h_n) .$$

Now right-translation is always a homeomorphism (and in ordinary differential geometry it also always is a diffeomorphism), $V = U \cdot g$ is an open neighbourhood of g . But any open set is saturated with respect to nilpotent parts, implying in particular that we have the equality $U \cdot g = U \cdot (\mathbf{B}g)$. The important observation now is that when

we fix, either in the multiplication or in the left-action, to an element of $\mathbf{B}G$, the function will be smooth in the remaining variable. This means that the maps $\Phi(\mathbf{B}h_i, \cdot) : M \rightarrow M$ and $R_{\mathbf{B}g} : G \rightarrow G$ are smooth. The group property of a left-action then tells us that the map $\Phi(\mathbf{B}g, \cdot) : M \rightarrow M$ is smooth, as we have

$$\Phi(\mathbf{B}g, m) = \Phi(\mathbf{B}h_1, \Phi(\mathbf{B}h_2, \dots \Phi(\mathbf{B}h_n, m) \dots)) ,$$

which is the composition of the smooth maps $\Phi(\mathbf{B}h_i, \cdot)$. It now suffices to note that the restriction of Φ to $V \times M$ equals the composition of the following smooth maps $\Psi : V \times M \rightarrow U \times M$, $(k, m) \mapsto (R_{\mathbf{B}g}k, \Phi(\mathbf{B}g, m))$ and $\Phi : U \times M \rightarrow M$, simply because we have the equality

$$\Phi(k, m) = \Phi(k(\mathbf{B}g)^{-1}, \Phi(\mathbf{B}g, m)) .$$

This shows that for any $g \in G$ there exists an open neighbourhood V of g such that Φ is smooth on $V \times M$, which implies that Φ is globally smooth. \square QED

3.11. Lemma. *Let G be a connected (super) Lie group with (super) Lie algebra \mathfrak{g} , let M be a (super) manifold and let $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a smooth representation of \mathfrak{g} on M . Then there exists at most one smooth left-action of G on M such that the vector fields $\rho(X)$ are the fundamental vector fields X^M .*

Proof. Let $\Psi_1, \Psi_2 : G \times M \rightarrow M$ be two such actions. It follows that the flow of the vector field Z_A is given by

$$\Phi_A(t, X, g, m) = (X, \exp(tX), \Phi_i(\exp(tX), m)) .$$

It follows immediately that Φ_1 and Φ_2 coincide on $\exp(\mathfrak{g}_0) \times M$. But $\exp(\mathfrak{g}_0)$ contains an open neighbourhood of the identity $e \in G$ and for a connected topological group, an open neighbourhood generates the whole group. The group property $\Psi_i(s, \Psi_i(t, m)) = \Psi(s + t, m)$ of a left-action then implies that Φ_1 and Φ_2 coincide on the whole of $G \times M$. \square QED

3.12. Theorem. *Let G be a connected and simply connected (super) Lie group with (super) Lie algebra \mathfrak{g} , let M be a smooth (super) manifold and let ρ be a smooth representation of \mathfrak{g} on M . If all vector fields $\rho(X)$, $X \in \mathbf{B}\mathfrak{g}_0$ are complete, then there exists a unique smooth left-action of G on M such that the vector fields $\rho(X)$ are the fundamental vector fields X^M .*

Proof. We start with a set-theoretic definition of the action $\Phi : G \times M \rightarrow M$. We thus choose $(g, m) \in M$ and we want to define $\Phi(g, m) \in M$. As the vector fields $\rho(X)$, $X \in \mathbf{Bg}_0$ are complete, the projection $p : G \times M \rightarrow G$ is a covering map when we equip $G \times M$ with the leaf-topology [3.8]. As G is connected, the restriction to the leaf-connected component $L_{(e,m)} \subset G \times M$ containing (e, m) thus is a covering map too. As G is simply connected, it must be a homeomorphism and in particular a bijection. Hence there exists a unique $m' \in M$ such that $(g, m') \in L_{(e,m)}$. We then define $\Phi(g, m) = m'$:

$$\Phi(g, m) = m' \iff (g, m') \in L_{(e,m)} .$$

To show that this is a left-action, choose $h \in G$. Then, as for g , there exists m'' such that $(h, m'') \in L_{(e,m')}$:

$$(3.13) \quad \Phi(g, m) = m' , \Phi(h, m') = m'' \iff (g, m') \in L_{(e,m)} , (h, m'') \in L_{(e,m')} .$$

According to [3.9] $R_g(L_{(e,m')})$ is a leaf-connected component. Moreover, it contains $(g, m') = R_g(e, m')$. Hence $R_g(L_{(e,m')}) = L_{(e,m)}$. But $R_g(h, m'') \in R_g(L_{(e,m')}) = L_{(e,m)}$ and thus

$$(3.14) \quad (hg, m'') \in L_{(e,m)} \iff \Phi(hg, m) = m'' = \Phi(h, m') = \Phi(h, \Phi(g, m)) .$$

As we obviously have $\Phi(e, m) = m$, we have shown that the map Φ is a left-action of G on M .

In order to show that Φ is a smooth action, we want to apply [3.10]. We claim that the neighbourhood U on which the exponential map is a diffeomorphism will do. More precisely, let $V \subset \mathfrak{g}_0$ be an open neighbourhood of 0 and let $U \subset G$ be an open neighbourhood of $e \in G$ such that $\exp : V \rightarrow U$ is a diffeomorphism. We then define the map $\psi : U \times M \rightarrow G \times M$, using [3.3], by

$$\begin{aligned} \psi(g, m) &= p_{23}(\Phi_A(1, \exp^{-1}(g), e, m)) \\ &= (\exp(1 \cdot \exp^{-1}(g))e, \Psi_M(1, \exp^{-1}(g), m)) \\ &= (g, \Psi_M(1, \exp^{-1}(g), m)) , \end{aligned}$$

which is smooth because Ψ_M is smooth and thus the restriction with $t = 1$ fixed is also smooth (for super manifolds this is because 1 has real coordinates, or equivalently, belongs to the body of \mathcal{A}_0). The subset of $\{0\} \times G \times M \subset \mathfrak{g}_0 \times G \times M$ is a submanifold, also in the super differential geometric context, and the restriction of Z_A to this

submanifold is zero. Its flow thus is the constant map, and hence via the canonical inclusion, we also have

$$\Phi_A(t, 0, g, m) = (0, g, m)$$

for all (t, g, m) and thus in particular

$$\psi(e, m) = (e, \Psi_M(1, 0, m)) = (e, m) .$$

We now note that for fixed $m \in M$ the map $\psi_m : U \rightarrow G \times M$ defined as

$$\psi_m(g) = \psi(g, m) = p_{23}(\Phi_A(1, \exp^{-1}(g), e, m))$$

is leaf-continuous as the composition of the (smooth and hence) continuous map \exp^{-1} with the leaf-continuous map Φ_A [3.6] and with the leaf-continuous projection p_{23} . As U is connected, its image is contained in a leaf-connected component. As $\psi_m(e) = (e, m)$, we thus must have $\psi_m(g) \in L_{(e, m)}$. In other words, we have shown

$$\psi(g, m) = (g, \Psi_M(1, \exp^{-1}(g), m)) \in L_{(e, m)} ,$$

which is by definition of the flow Φ equivalent to

$$\Psi_M(1, \exp^{-1}(g), m) = \Phi(g, m) .$$

It follows that Φ restricted to $U \times M$ is the smooth map $\Psi_M(1, \cdot)$, showing that the condition of [3.10] is satisfied, and thus Φ is a smooth left-action of G on M . Uniqueness follows from [3.11]. \square

3.15. Remark. In the ordinary differential geometric context we could have foregone the use of the leaf topology altogether. Let us outline how to proceed. We do need the foliation \mathcal{F} and the vector fields Z_G , Z_M and Z_A as well as their flows. But we do not need [3.2] or [3.6]. To prove [3.8] we start with the observation that through any point passes a leaf L , i.e., a maximal integral manifold for the foliation, which is an immersed submanifold. To prove that the projection $p : G \times M \rightarrow G$ is a covering map when restricted to L , we have to show in particular that it is surjective, for which we (also) need that the flow of Z_A is complete. To prove surjectivity, let $g_o \in \overline{p(L)}$ be a point in the closure of the image, and let $U \subset G$ be a connected open neighbourhood of the identity on which the exponential map is a diffeomorphism. Then there exists $g \in g_o \cdot U^{-1} \cap p(L)$, and thus there exists $m \in M$ such that $(g, m) \in L$ and $g_o \in g \cdot U$. As the flow of Z_A is complete, we can apply the smooth map ψ_m of the proof of [3.8] to obtain a map from the open neighbourhood $g \cdot U$ to $G \times M$. This map is tangent to the foliation, and hence (as it is connected) its image is included in a leaf, which must be L . As $p \circ \psi_m = id$, this shows that

$g \cdot U$ is contained in $p(L)$ and thus g_o , which belongs to the closure of $p(L)$ admits the open neighbourhood $g \cdot U$ contained in $p(L)$. It follows that $p(L)$ is both open and closed, hence it must be equal to G , i.e., p is surjective when restricted to a leaf L .¹ And then we can use exactly the same reasoning as in the proof of [3.8] to prove that $p : L \rightarrow G$ is a covering map.

We do not need the full statement of [3.9], only the particular case that R_h maps leaves of \mathcal{F} to leaves. But this is a direct consequence of the fact that \mathcal{F} is invariant under these right-translations. The proofs of [3.10], [3.11] and [3.12] remain unchanged (and the notation in the proof of [3.10] simplifies by omitting the body map). We needed the leaf topology in the super differential geometric context because there it is not guaranteed that there exists a leaf through (e, m) when m is not in the body of M . The connected components of the leaf topology provide us with these “leaves,” but we can no longer use differential geometric arguments.

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¹This argument is also used in the proof of [3.12], but there it is hidden in the statement that the restriction of a covering map to a connected component is again a covering map.